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On a one-dimensional system associated with a gl($m|n$) vertex model

Junji Suzuki

Institute of Physics, College of Arts and Sciences, University of Tokyo, Komaba 3-8-1, Meguro-ku, Tokyo, Japan

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Abstract. A one-dimensional (1D) system associated with a gl($m|n$) vertex model is investigated. We add a chemical potential term so that the model reduces to the supersymmetric t–J model for $(m,n) = (2,1)$. It is conjectured that the ground state of the model possesses SU(m) symmetry in the rational limit. The observation by Kawakami and Yang on the t–J model is extended to a more general theorem of the 1D field theory: The t–J model is just an example of the present model, and such systems can be described by a multi-component Gaussian model compactified on tori with different radii.

1. Introduction

Supersymmetric models in low dimensions are providing current theoretical problems. To give an example, the representation theory for quantization of a Lie superalgebra [1] is a subject of interest. The universal R-matrix of osp(2|1) is discussed by Kulish and Reshetikhin [2]. Saleur [3] obtained the osp(2|1) R-matrix with spectral parameter by generalizing the method developed by Drienfeld [4] and Jimbo [5]. He found that it is essentially given by the Boltzmann weight for the 19-vertex model (or the SU(2) level 2 Wess–Zumino–Witten model). Deguchi and Akutsu [6] obtained the link invariant from the gl($m|n$) model. Deguchi and Fujii [7], and Okado [8] have recently solved the IRF model associated with sl($m|n$).

The mathematical structure underlying supersymmetric models has been unveiled by these works. It seems, however, that their physical nature remains unclear.

In this article, we present some numerical and analytical studies of a quantum one-dimensional (1D) system associated with a gl($m|n$) vertex model in two dimensions [9–12]. We study the limiting case of a chemical potential term so that the model reduces to the supersymmetric t–J model [13–16] for $(m,n) = (2,1)$†. It is found that the ground state possesses SU(m) symmetry. Our result suggests that the model can be described by a multi-component Gaussian system compactified on tori with different radii. This is a straightforward generalization of the observation by Kawakami and Yang [16].

This paper is organized as follows. In the next section, we introduce the gl($m|n$) vertex model on a square lattice and its 1D counterpart solved by Perk and Schultz

† The term 'supersymmetric' is somewhat misleading. The Hamiltonian for the solvable t–J model does not commute with the action of gl(2|1) as pointed out by Bares and Blatter [15].

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[9–12]. The definition of the model and known results are reviewed. We present the results of the direct diagonalization of the Hamiltonian in the 'rational' case in section 3. From these data, we conjecture the ground state configuration. We analyse the low excitations near the ground state by making use of the Bethe ansatz equation (the BAE, for short). We also interpret this model in terms of field theory. In the last section, we present several open questions arising from this study .

2. The model and known results

Several years ago, Perk and Schultz [9–12] solved a gl$(m|n)$ vertex model on a square lattice and its 1D counterpart. This section is devoted to a simple explanation of their models. We assign edge variables $\mu = 1, \ldots, q+1, (q+1 = m+n)$ to each horizontal and vertical bond on a square lattice. Possible configurations at a vertex are given in figure 1, and the corresponding vertex weights are given by

$$(f(\lambda - \epsilon_\alpha u), G_{\alpha,\beta} f(u), f(\lambda) e^{g(u)}) \qquad \alpha \neq \beta \qquad \alpha, \beta = 1, \ldots, q+1 \tag{1}$$

where

$$f(u) = \sin(u) \qquad \text{or} \qquad \sinh(u)$$

$$g(u) = iu \frac{\operatorname{sgn}(\alpha - \beta) - 2(\alpha - \beta)}{(q+1)} \qquad \text{or} \qquad u \frac{\operatorname{sgn}(\alpha - \beta) - 2(\alpha - \beta)}{(q+1)}$$

and $\epsilon_\alpha = \pm 1$. $G_{\alpha,\beta}$ are arbitrary constants satisfying $G_{\alpha,\beta} G_{\beta,\alpha} = 1$. Here we adopt slightly simpler vertex weights than the original ones.

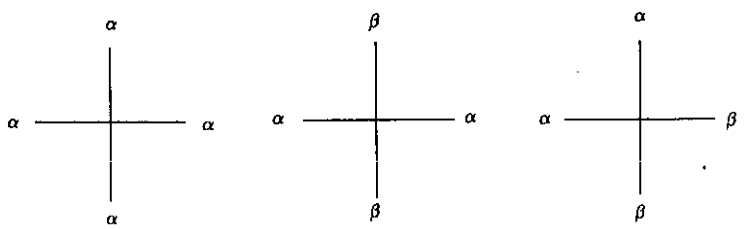

Figure 1. Allowed vertex configurations.

In the following, we assume that the first m ϵ's are 1 and the remainder -1. A periodic boundary condition is also assumed. We denote the number of vertical bonds with colour α by n_α. As is immediately seen, the $n_\alpha (\alpha = 1, \ldots q+1)$ are conserved quantities. Thus we can specify a state by $(n_1, \ldots, n_{q+1})$.

A solvable 1D Hamiltonian associated with this vertex model is given by [11]

$$\mathcal{H} = \sum_i \left(\sum_\alpha \epsilon_\alpha k(\lambda) e^i_{\alpha,\alpha} e^{i+1}_{\alpha,\alpha} + \sum_{\alpha \neq \beta} (-G_{\alpha,\beta} e^i_{\alpha,\beta} e^{i+1}_{\beta,\alpha} + \omega_{\alpha,\beta}(\lambda) e^i_{\alpha,\alpha} e^{i+1}_{\beta,\beta}) + \sum_\alpha h_\alpha e^i_{\alpha,\alpha} \right) \tag{2}$$

where $e_{\alpha,\beta}$ denotes the matrix element; $\{e_{\alpha,\beta}\}_{a,b} = \delta_{a,\alpha}\delta_{b,\beta}$, the superscript '$i$' represents the position in the chain and

$$\omega_{\alpha,\beta}(\lambda) = \mathrm{i}\sin(\lambda)\frac{\mathrm{sgn}(\alpha-\beta)-2(\alpha-\beta)}{(q+1)} \qquad \text{or}$$
$$\sinh(\lambda)\frac{\mathrm{sgn}(\alpha-\beta)-2(\alpha-\beta)}{(q+1)}$$
$$k(\lambda) = \cos(\lambda) \qquad \text{or} \qquad \cosh(\lambda).$$

h_α are arbitrary numbers, which will be referred to as the chemical potential.

The BAE for this system reads [12] as

$$\left(\epsilon_1\frac{f(\lambda/2-\mathrm{i}\epsilon_1 x_j^1)}{f(\lambda/2+\mathrm{i}\epsilon_1 x_j^1)}\right)^N = \prod_k \epsilon_2\frac{f(\lambda/2-\mathrm{i}\epsilon_2(x_k^2-x_j^1))}{f(\lambda/2+\mathrm{i}\epsilon_2(x_k^2-x_j^1))}\prod_k\frac{f(\lambda-\mathrm{i}\epsilon_2(x_j^1-x_k^1))}{f(\lambda+\mathrm{i}\epsilon_1(x_j^1-x_k^1))}$$

$$\prod_k \epsilon_\beta\frac{f(\lambda/2-\mathrm{i}\epsilon_\beta(x_j^\beta-x_k^{\beta-1}))}{f(\lambda/2+\mathrm{i}\epsilon_\beta(x_j^\beta-x_k^{\beta-1}))} = \prod_k \epsilon_{\beta+1}\frac{f(\lambda/2-\mathrm{i}\epsilon_{\beta+1}(x_k^{\beta+1}-x_j^\beta))}{f(\lambda/2+\mathrm{i}\epsilon_{\beta+1}(x_k^{\beta+1}-x_j^\beta))}$$
$$\times\prod_k\frac{f(\lambda-\mathrm{i}\epsilon_{\beta+1}(x_j^\beta-x_k^\beta))}{f(\lambda+\mathrm{i}\epsilon_\beta(x_j^\beta-x_k^\beta))} \tag{3}$$

$$\prod_k\frac{f(\lambda-\mathrm{i}\epsilon_{q+1}(x_j^q-x_k^q))}{f(\lambda-\mathrm{i}\epsilon_q(x_j^q-x_k^q))} = \prod_k \epsilon_q\frac{f(\lambda/2-\mathrm{i}\epsilon_q(x_j^q-x_k^{q-1}))}{f(\lambda/2+\mathrm{i}\epsilon_q(x_j^q-x_k^{q-1}))}$$
$$\times\prod_k\frac{f(\lambda-\mathrm{i}\epsilon_q(x_k^q-x_j^q))}{f(\lambda+\mathrm{i}\epsilon_q(x_k^q-x_j^q))}$$

where N represents the linear size of system. The second equation is valid for $\beta = 2,\ldots,q-1$. Here the x_j^α are related to quasi momenta, and are given by the roots of this BAE. In terms of x_j^1, we can represent the energy per site as

$$e = \epsilon_1 f'(\lambda) + \frac{\epsilon_1}{N}\sum_j\left(\frac{f'(\lambda/2-\mathrm{i}\epsilon_1 x_j^1)}{f(\lambda/2-\mathrm{i}\epsilon_1 x_j^1)} + \frac{f'(\lambda/2+\mathrm{i}\epsilon_1 x_j^1)}{f(\lambda/2+\mathrm{i}\epsilon_1 x_j^1)}\right) + \frac{1}{N}\sum_{\alpha=1}^{q+1} n_\alpha h_\alpha. \tag{4}$$

Note that the BAE and the energy is independent of $G_{\alpha,\beta}$. We also remark that the number of roots N_α or, equivalently, the numbers of x_j^α are related to n_α defined previously by

$$\begin{aligned} &N - N_1 = n_1 \qquad N_1 - N_2 = n_2,\ldots \\ &N_{q-1} - N_q = n_q \qquad N_q = n_{q+1}. \end{aligned} \tag{5}$$

In principle, all eigenvalues are obtained by solving this BAE. Perk and Schultz [11] found that the ground state is frozen if $f(u) = \sinh(u)$ and $h_\alpha = 0$ for $\alpha = m+1,\ldots m+n$. The ground state has a ferromagnetic-like configuration in this case: $n_j = N$ for j such that $\epsilon_j = 1$ and $h_j = \max h_\alpha$.

We do not know, however, how to locate the roots of the BAE for the ground state in general. We avoid making assumptions on the distribution of the roots of the BAE. Instead, we diagonalize numerically the Hamiltonian (2) for small chains applying the method developed by Householder and Lanczös.

3. Results of numerical diagonalization

In this section, we study the rational case $u \to u\lambda, \lambda \to 0$. There is some arbitrariness in determining the explicit forms for h_α in the last term of the Hamiltonian (2). We fix it in the following way. For $(m,n) = (2,1), (\epsilon_1, \epsilon_2, \epsilon_3) = (1,1,-1)$ and we regard edge variables $1,2,3$ as the up-electron, down-electron and hole, respectively. The supersymmetric t–J model has the same matrix element as our model if we choose

(i) $h_\alpha = -\epsilon_\alpha$
(ii) $\lambda \to 0$
(iii) $G_{1,2} = G_{2,1} = -1$ and other $G_{\alpha,\beta}$ to be 1.

Then the Hamiltonian is [17]

$$\mathcal{H}' = \sum_i \sum_{\alpha \neq \beta} e^i_{\alpha,\beta} e^{i+1}_{\beta,\alpha} (-1)^{F(\beta)} \tag{2'}$$

where $F(\beta) = 0,0,1$ for $\beta = 1,2,3$ respectively†. As a check we have compared the ground-state energy of this model with that of the XXX model. We verify that they indeed agree and will converge to $-2\log 2$ [13, 14] in the infinite lattice limit.

In the following, we consider the rational limit of the model equation (2'), but the index α runs as $1, \ldots, m+n$.

We present our results for $(m,n) = (2,1)$ and $(3,1)$ in figures 2(*a*) and (*b*) and tables 1 and 2.

Table 1. Explicit values of energies per site for the gl$(2|n)$ model. We present the values of the lowest energies among 'spin' (or 'charge') excited states in the corresponding columns.

System size	Ground-state energy	Spin excitation	Charge excitation
4	−1.500 000 0	−1.000 000 0	−1.207 106 8
6	−1.434 258 5	−1.206 011 3	−1.271 202 1
8	−1.412 773 4	−1.282 104 8	−1.301 895 3
10	−1.403 089 3	−1.318 441 5	−1.319 789 3
12	−1.397 898 5	−1.338 590 6	−1.331 474 6

Table 2. Explicit values of energies per site for gl$(3|n)$ model.

System size	Ground-state energy	Spin excitation	Charge excitation
3	−2.000 000 0	−1.333 333 3	−1.333 333 3
6	−1.767 591 9	−1.604 535 5	−1.498 502 6
9	−1.731 048 4	−1.658 008 7	−1.563 870 8
12	−1.718 710 6	−1.677 335 4	—

† The author was informed from Dr T Deguchi that this observation was also given by Dr S Nakaya (1990, unpublished).

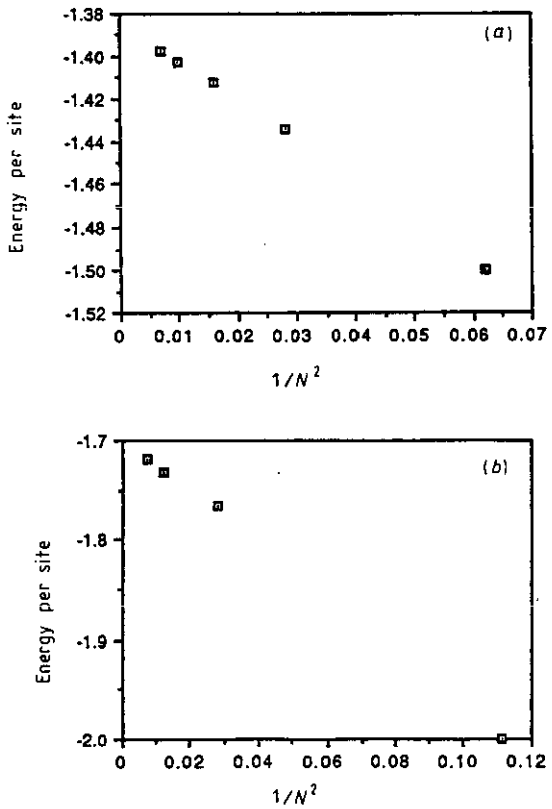

Figure 2. Ground-state energies per site against $1/N^2$ for the rational gl$(m|n)$ chain: (*a*) $(m,n)=(2,1)$; (*b*) $(3,1)$.

The ground-state energy per site at finite size N should behave as

$$e_N = e_\infty - \frac{\pi v c}{6N^2} + o\left(\frac{1}{N^2}\right) \tag{6}$$

for a system with conformal invariance [19, 20]. We have, therefore, plotted the result of the calculation choosing $1/N^2$ as the horizontal axis.

We find numerically that the first few excited energies of the $(2,2)$ and $(2,3)$ model agree with those of the $(2,1)$ model. Similarly, the low excitation energies of the $(3,2)$ model coincide with those of the $(3,1)$ model.

These results suggest that for the system $(m,n)=(l>1,k)$ with its linear size lp the ground state is given by $(p,\ldots,p,0,\ldots,0)$. (Hereafter we call the states 'half-filling' if $n_1+\cdots+n_l=lp$, and charge excited states if $n_1+\cdots+n_l<lp$.) The result for the ground state is natural and gives an explanation as to why the supersymmetric t–J model behaves in a very similar way to the Heisenberg model at half filling. The symmetry inherited in the model is gl$(2|1)$; however this reduces to SU(2) because of the symmetry-breaking term.

These results can be generalized to the assertion that gl$(m|n)$ is reduced to SU(m) at least for the ground state. This is the main result in this article. The excited spectra reflect the full SU$(m)\times$SU(n) symmetry of the Hamiltonian (2′).

We also calculated the dispersion relations of elementary excitations for the $(2,1)$ model ($N = 6$) and for the $(3,1)$ model ($N = 9$). We estimate the sound velocities from these data, given in table 3. Then formula (6) yields the central charges. Taking account of logarithmic corrections, we estimate that the central charges for the $(2,n)$ and $(3,n)$ models are 1 and 2, respectively.

Table 3. Sound velocities and central charges for $(2,1)$ and $(3,1)$ models.

	Sound velocity	c	$c(N \to \infty)$
$(2,1)$	−3.443 058	−1.038...	1
$(3,1)$	−2.252 052	−2.231...	2

4. Analyses through the Bethe ansatz equation

Having identified the ground state, we can analyse the low excitations by using equation (3) and standard techniques for dealing with the BAE. We first recall known results for $(m,n) = (2,1)$.

Bares and Blatter [14] started with another BAE which takes a different form from the present one, although these two approaches are equivalent. They found that the spin excitation should be described by the des Cloizeaux–Pearson mode at half-filling. Away from half-filling, both charge and spin excitation become massless. They can be regarded as two Gaussian fields compactified on tori with different radii [16], which is a common feature of the Luttinger liquids [20].

BAE (3) gives a more natural explanation for the Bares and Blatter result. It is clear that the BAE for $(\epsilon_1, \epsilon_2, \epsilon_3) = (1,1,-1)$ is equivalent to that for the XXX model if the number of roots is $N_1 = N/2, N_2 = 0$. Note that the present model and the XXX model have the same expression for the energy in terms of $\{x_j^1\}$. Therefore it is apparent that the BAE leads to their result more straightforwardly.

Then let us consider the case $m > 2$. As a natural generalization, we identify m components as 'particles' and n components as 'vacancies'. It is now clear that at half-filling the BAE is equivalent to that for the SU(m) model. Then we can utilize the results of the standard method of finite-size correction [21–23]. The central charge for the SU(m) model is equal to its rank, $m-1$. This is consistent with our data in table 3 for the $(2,1)$ and $(3,1)$ models. The scaling dimensions can also be derived as long as we are concerned with 'spin' excitations. Let $\mathcal{C}$ be the Cartan matrix for A_{m-1} type Lie algebra. We furthermore introduce two vectors r and s that have integer components. They represent the indices of the excitations. Then the scaling dimension Δ has form

$$\Delta = \frac{1}{g} r\mathcal{C}r + \frac{g}{4} s\mathcal{C}^{-1}s \tag{7}$$

with $g = 4$ at the rational fixed point. The two vectors, r and s, have been interpreted as the generalized spin and vorticity indices.

In the half-filling case, we can easily verify that the dispersion relation for the charge excitation is not linear. It therefore breaks the Lorentz invariance. Suppose

that the symmetry-breaking term (namely the chemical potential term) is given by $h_\alpha = -h\epsilon_\alpha$ and $0 < h < 1$. There should be a critical value h_c below which the ground state is no longer given by the SU(m) symmetric state but is different from half-filling. We then expect that both the 'spin' and 'charge' excitations will be massless [16] below h_c. These excitation energies are determined by the dressed energy function formulation [24]. The critical potential is estimated as follows. Let us write the chemical potential term as $\sum h_\alpha e_{\alpha,\alpha}$. Then the dressed energy functions are given by the solution of the following integral equations.

$$E_\alpha(x) = -E_\alpha^{(0)}(x) - \sum_\beta \int_{-\Lambda_\beta}^{\Lambda_\beta} \frac{T_{\alpha,\beta}(x-x')E_\beta(x')}{2\pi}\,\mathrm{d}x' \qquad \alpha = 1,\ldots,q \tag{8a}$$

where $T_{\alpha,\beta}$ are defined by

$$\begin{aligned} &T_{i-1,i}(x) = -\psi'(x,\tfrac{1}{2}) \qquad i = 2,\ldots,q;\ i \neq m+1 \\ &T_{m,m+1}(x) = \psi'(x,\tfrac{1}{2}) \\ &T_{i,i}(x) = \psi'(x,1) \qquad i = 1,\ldots,q;\ i \neq m \end{aligned} \tag{8b}$$

and $T_{i,j} = 0$, otherwise. $E_\alpha^{(0)}$ signify

$$\begin{aligned} &E_1^{(0)} = \psi'(x,\tfrac{1}{2}) + h_1 - h_2 \\ &E_i^{(0)} = h_i - h_{i+1} \qquad i = 2,\ldots,q \end{aligned} \tag{8c}$$

and $\psi(x,y)$ is given by

$$\psi(x,y) = \mathrm{i}\log\left(-\frac{x+\mathrm{i}y}{x-\mathrm{i}y}\right). \tag{8d}$$

Furthermore we demand $E_\alpha(\Lambda_\alpha) = 0$, which is equivalent to the condition of a ground state with given fields.

Remember that $\{h_i\}$ is given by

$$(h_1, h_2, \ldots, h_m, h_{m+1}, \ldots, h_{q+1}) = (-h, -h, \ldots, -h, h, \ldots, h) \tag{9}$$

in the present case. For $h = 1$, $E_\alpha^{h=1}$ is given by

$$E_\alpha^{h=1}(x) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx} \frac{\sinh k(m-\alpha)/2}{\sinh km/2}\,\mathrm{d}k \qquad \alpha = 1,\ldots,m-1 \tag{10a}$$

$$E_\alpha^{h=1}(x) = 0 \qquad \alpha = m,\ldots,q. \tag{10b}$$

We consider equation (8a) for $\alpha = m$. It is reasonable to assume that Λ_{m+1} is infinitesimally small if $h \sim h_c$. We therefore neglect the contribution from the term containing an integral over $[-\Lambda_{m+1}, \Lambda_{m+1}]$. We also assume that $E_\alpha(x)$ has minima at $x = 0$ in the interval $[-\Lambda_\alpha, \Lambda_\alpha]$. Then the condition for the critical potential is given by

$$E_m(x=0) = 0. \tag{11}$$

This leads to a conclusion

$$h_c = \frac{1}{2}\int_{-\infty}^{\infty} E_{m-1}^{h=1}(x)\frac{\psi'(x,\frac{1}{2})}{2\pi}\,\mathrm{d}x. \tag{12}$$

For $m = 2$, the right-hand side of equation (12) is numerically given by 0.693 147. On the other hand, assuming the scaling form, h_c (for size N chain) $\sim h_c$ (for infinite system) + constant/N^2†, we obtain a crude estimation from the finite chain

† This extrapolation technique was suggested to the author by Professor V Rittenberg.

Table 4. Critical values of chemical potential for given system sizes.

System size	h_c
6	−0.598 57
8	−0.634 02
10	−0.653 04
12	−0.664 09

calculation (table 4), $h_c \sim 0.686$.

This is close to the result derived from the earlier assumptions. This fact supports the validity of our assumptions.

Let us present the interpretation of the model in terms of field theoretical language, before closing this section.

Judging from the scaling dimensions (7), we expect that the effective field theory underlying this model at half-filling will be described by the Gaussian field compactified on A_{m-1} torus. Such a field theory has been proposed by Kostov [25], Fateev and Lukyanov [26]. Let Λ_i $(i = 1, \ldots, m-1)$ be fundamental weights for A_{m-1}, and let ϕ_i be Gaussian fields. We consider a field Φ given by $\sum_{i=1}^{m-1} \phi_i \Lambda_i$. Then the field theoretical version of the present model at half-filling is described by the action

$$\mathcal{A} = \frac{\pi g}{2} \int (\partial \Phi)^2 \, \mathrm{d}^2 x. \tag{13}$$

Away from half-filling, we should also consider charge excitations and many other composite excitations. Their excitation spectra and other thermodynamic properties are determined by solving equation (8*a*) at arbitrary filling. Rigidities to various excitations, namely the Fermi velocities, are given by [24]

$$v_\alpha = \frac{E'_\alpha(\Lambda_\alpha)}{2\pi \rho_\alpha(\Lambda_\alpha)} \qquad \alpha = 1, \ldots, q \tag{14}$$

where $\rho_\alpha(x)(\alpha = 1, \ldots, q)$ denote the densities of the roots of the BAE. The Fermi velocities depend on h_α. We thus expect that the effective field theory underlying the model away from half-filling also to be given by the multi-component Gaussian model but with different radii of compactifications

$$\mathcal{A} \sim \sum_{\alpha=1}^{q} v_\alpha \int (\partial \phi_\alpha)^2 \, \mathrm{d}^2 z. \tag{15}$$

This is the natural generalization of the observation by Kawakami and Yang [16] in the case of $(m, n) = (2, 1)$.

5. Future problems and conclusion

In this article, we have numerically and analytically studied 1D spin chains associated with the gl$(m|n)$ vertex model. We find that the symmetry of the ground state is SU(m) in the rational case. The solvable t–J model is an example of the present

model. Thus we have succeeded in generalizing the results of the t–J model, gl(2|1), to the gl($m|n$) cases.

We present two open questions which we have found in our numerical calculations in the end.

(i) Throughout the arguments in sections 3 and 4 we have assumed $m > 1$. We also performed numerical calculations on the $(m, n) = (1, q)$ model (figure 3, table 5).

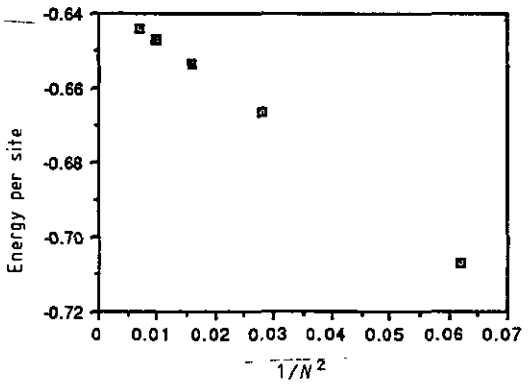

Figure 3. Ground-state energies per site against $1/N^2$ for the rational gl(1|2) chain. We have numerically verified that the ground-state energy is given by the same values for gl($1|n$).

Table 5. Ground energy per site of the gl($l|m$) model.

System size	Ground-state energy
4	−0.707 106 8
6	−0.666 666 7
8	−0.653 281 5
10	−0.647 213 6
12	−0.643 950 6

It is well known that the model reduces to the free fermion one in the case of $(1,1)$. Although we have not given any details of the data, the lowest energy for any q is the same as that for $q = 1$. When $q \geqslant 2$, the lowest energy states are degenerate in each case of q. In fact a set of quantum numbers that satisfies $n_1 = \sum_{j>1} n_j = N/2$ can lead to the ground-state energy. This is what we have found by numerical calculations, which are, as yet, not very general. Hence we can only state this proposition as a conjecture.

(ii) The Perk–Schultz model with $f(x) = \sin(x)$ and $h_\alpha = 0 (\alpha = 1, \ldots, q+1)$ is now a problem of current interest [27–29]. It is interesting to see whether the main results of the present paper (obtained for the rational case) hold true in the trigonometric case. Our model contains the chemical potential as an important factor. Hence we include the chemical potential term

$$-\cos\lambda \sum_{i,\alpha} \epsilon_\alpha e^i_{\alpha,\alpha}. \tag{16}$$

This chemical potential reduces to the previous one ($h = 1$) in the rational limit. Moreover, it is invariant under the simultaneous transformations: $\lambda \to \pi - \lambda, \epsilon_\alpha \to -\epsilon_\alpha$. We should remark that our choice is *not* the only possible form that satisfies these two conditions.

In the case of the $(m, n) = (2, 1)$ model, we have found that the ground state has SU(2) symmetry for $\lambda = \pi/l, l = 4 \sim 8$.

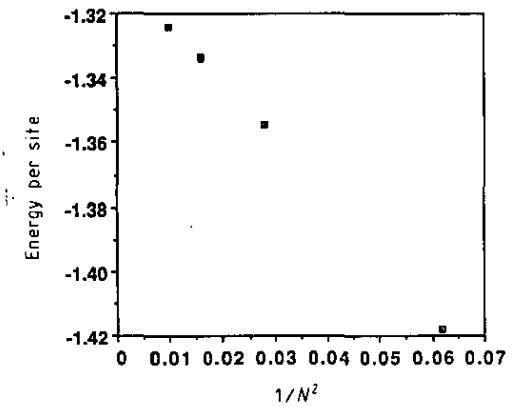

Figure 4. Ground-state energies per site against $1/N^2$ for the trigonometric gl(2|1) chain with chemical potential term (17). The coupling constant λ is chosen to be $\pi/7$.

It is, however, impossible to tell which is the ground state for $\lambda \simeq \pi/2$ because of the smallness of the system. We cannot disregard the possibility that the true ground state is incommensurate with the lattice for the system size with which we can deal. (This is the case for the Hubbard model in a magnetic field [30].) It is, however, difficult to deal with a larger sized system by the direct matrix diagonalization method, since the Hamiltonian is not Hermite. Further careful studies, possibly by analyses of the BAE, are needed to make a conclusive statement in the trigonometric case.

We hope to discuss these questions in future publications.

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